Preordered topological spaces, utility and jointly utility functions

Alessandro Caterino

A topological preordered space is a triple $(X, \tau, \preceq)$ where X is a set endowed with a topology τ and a preorder $\preceq$. The preorder $\preceq$ is said to be continuous (or semi-closed) if for each $a \in X$ the sets $[a, +\infty[= \{x \in X : a \preceq x\}$ and $]-\infty, a] = \{x \in X : x \preceq a\}$ are τ-closed. The preorder $\preceq$ is said to be *closed* if its graph $\{(x, y) \in X \times X \ : \ x \preceq y\}$ is closed in $X \times X$.

A real function $f : X \to \mathbb{R}$ is isotone if $x \preceq y$ implies $f(x) \leq f(y)$ for every $x, y \in X$ and it is a utility function if it is isotone and $x \prec y$ implies $f(x) < f(y)$. A family $\mathcal{U} \subset C(X, \mathbb{R})$ (multi-utility) represents the preorder $\preceq$ if $[x \preceq y$ if and only if $\forall f \in \mathcal{U}, f(x) \leq f(y)]$. In a similar way, if R is a binary relation on X, one can define a utility representation of R

Classical theorems of Eilenberg and Debreu state that every continuous total order, defined respectively on a connected separable space and on a second countable space, have a continuous representation. The notions of network and netweight provide useful tools in the theory of representation. In [8], by using these concepts, it is proved the following result that includes the above mentioned results.

Theorem 1 ([8]) *Let $(X, \tau, \preceq)$ be a preordered topological space where $\preceq$ is a continuous total preorder. If $X = \bigcup_{i \in I\!N} X_i$, where every X_i is connected separable or has countable network, then $\preceq$ has a continuous representation.*

In [2] the existence of a continuous utility function for a binary relation is characterized. In particular, the following result is proved.

Theorem 2 ([2]) *Let X be a topological space with countable network and let R be a continuous binary relation on X. Then R has a continuous utility representation.*

The utility representation problem was solved by Levin (1983) for non linear closed preorders defined in locally compact second countable spaces.

A generalization of Levin's Theorem to submetrizable k_ω-spaces is presented in [7] and, by using a different technique, it is also proved in [2].

The k_ω-spaces have nice properties. In [10] the author proves that every k_ω-space, equipped with a closed preorder, is normally preordered and that every second countable normally preordered space has a countable utility (multi-utility) representation.

More generally, in [3] it is proved that every normally preordered space with countable netweight has a countable utility representation. Therefore, every submetrizable k_ω-space has a countable utility representation too.

If X is a topological space and Γ is a set of closed preorders on X, the problem of the existence of jointly continuous utility functions is to find topological conditions on Γ and X in order that there exists a continuous function

$$u : \Gamma \times X \to \mathbb{R}$$

such that $u(\preceq, \cdot)$ is a utility function for every $\preceq \in \Gamma$.

In [9] Levin proved some results on the existence of jointly continuous utility functions, for example by assuming that Γ and X are locally compact and second countable.

In [7] Levin's results have been generalized to some non-metrizable cases, for instance if Γ and X are submetrizable hemicompact spaces and $\Gamma \times X$ is a k-space.

Back in [1] revisited Levin's Theorems by using partial maps and hypertopologies.
He considered the set $\mathcal{P}$ of all linear closed preorders defined on closed subsets of a locally compact second countable space X endowed with the topology of closed convergence $\tau_{\mathcal{F}}$ and the set $\mathcal{U}$ of all continuous real partial maps defined on closed subsets of X with the τ_c topology, a generalization of the compact-open topology. He showed the existence of a continuous map $\nu : \mathcal{P} \to \mathcal{U}$ such that $\nu(\preceq)$ is a utility function for every $\preceq \in \mathcal{P}$.

Under the hypotheses that X is regular and submetrizable by a boundedly compact metric, in [6] we have associated to the space of preorders $\mathcal{P}$ a suitable space of preorders $\widetilde{\mathcal{P}}$ satisfying the hypotheses of Back's Theorem. So, by using Back's result, we proved the following theorem:

Theorem 3 ([6]) *Let (X, τ) be regular and submetrizable by a boundedly compact metric ρ. There exists an continuous map*

$$\nu : (\mathcal{P}, \tau(\mathcal{L})) \to (\mathcal{U}_\tau, \tau_c)$$

such that $\nu(\preceq)$ is a isotone function for $\preceq$, for every $\preceq \in \mathcal{P}$. Moreover, $\nu(\preceq)$ is a utility function if and only if the ρ-closed preorder $\Gamma(\preceq)$ satisfies the following property: if $a \prec b$ then $\neg\, b\Gamma(\preceq)a$

The topology on $\mathcal{U}$ is just the τ_c topology, while the topology $\tau(\mathcal{L})$ in $\mathcal{P}$, defined by mean of a suitable convergence structure, is reminiscent of the topology of closed convergence.

Another generalization of Back's theorem to submetrizable k_ω-spaces is presented in [5].

References

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