Recent progress in minimal groups

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The $\mathcal{P}$-closed and the $\mathcal{P}$-minimal spaces are an evergreen topic in general topology, starting from the celebrated 1924 Memoir of Alexandroff and Urysohn. (For a category $\mathcal{P}$ of topological spaces, $X \in \mathcal{P}$ is $\mathcal{P}$-*closed*, if X is closed in every $Y \in \mathcal{P}$ containing X as a subspace; $(X, \tau) \in \mathcal{P}$ is $\mathcal{P}$-*minimal*, if every topology $\tau' \leq \tau$ on X with $(X, \tau') \in \mathcal{P}$ coincides with τ [1].)

Let $\mathcal{P}$ be the category of Hausdroff topological groups. While it is easy to see that the $\mathcal{P}$-closed groups are simply the (largely studied) complete groups, the $\mathcal{P}$-minimal groups, usually called simply *minimal groups*, present a novelty both in topology and algebra. The class $\mathcal{M}$ of minimal groups contains all compact groups, so one expects $\mathcal{M}$ to have similar stability properties (e.g., stability with respect to taking closed subgroup, products and quotients). G. Choquet's question on whether $\mathcal{M}$ is closed under products triggered the pioneering paper [2], providing a negative answer to Choquet's question, as well as a series of examples of non-compact groups in $\mathcal{M}$ (other example were given in [5]). Answering questions of Arhangel′skiĭ and Pestov, Megrelishvili [3] proved that every topological group is a group retract of a minimal group (in particular, $\mathcal{M}$ is not closed under taking closed subgroup, quotients and even retracts). On the other hand, closed subgroups of minimal abelian groups are still minimal. Moreover, according to a deep result of Prodanov and Stoyanov [4], the abelian groups in $\mathcal{M}$ are subgroups of compact groups. This shows the strong impact of the algebraic structure on the properties of $\mathcal{M}$.

The talk will introduce will discuss various properties of the class $\mathcal{M}$, with particular emphasis on products.

References

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