Topological interpretations of fuzzy subsets*

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Abstract

We show that the classical definition of a fuzzy subset carries additional structures of a topological nature. We look at the concept of a fuzzy subset and its corresponding α-cuts from an alternative point of view: namely, a fuzzy subset may be interpreted as a nested topology on a crisp set of reference, called a universe. Several kinds of fuzzy subsets associated to this interpretation are analyzed. Other topologies induced by fuzzy subsets are considered, paying special attention to their relationship with total preorders defined on the universe.

A new look at the classical definition of a fuzzy subset, introduced by L.A. Zadeh in his seminal work, reveals topological and other structures that were hidden in the original definition. Let us consider the standard definition of a fuzzy subset X (of a set U) as a function $\mu_X : U \to [0,1]$, where U denotes a nonempty (crisp) set of reference, usually called universe. This approach is obviously equivalent to the definition of a fuzzy subset as a nested family $\{U_\alpha : \alpha \in [0,1]\}$ of crisp subsets of U, called the α-cuts. Here $U_\alpha = \{t \in U : \mu_X(t) \geq \alpha\}$, for each $\alpha \in [0,1]$. But a nested family of subsets of a given crisp set U immediately induces a topology that is also nested. Thus, the classical definition of a fuzzy subset could be (re)-interpreted as a nested topology of a certain kind defined on a given crisp set. Moreover, different kinds of fuzzy subsets would lead to different sorts of nested topologies.

We wonder about which additional information furnishes this new setting.

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There are important classes of topologies (e.g., the lower preorderable ones) that are, by definition, nested. Also, a key fact related to nested topologies is that of defining different kinds of orderings (e.g. linear orders, total preorders) on a given nonempty crisp set. Having this in mind, we will also analyze if those orderings can directly be defined by means of suitable fuzzy subsets.

Keywords: Fuzzy subsets; nested topologies; total preorders.

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