Lattices of continuous and uniformly continuous functions

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On characterizing innerly the set $C(X)$ of real-valued continuous functions on a topological space X depends essentially on the algebraic structure we are interested in.

Whenever $X \neq \varnothing$, the set $C(X)$ endowed with its pointwise defined order becomes a distributive lattice containing all the constant functions into $\mathbb{R}$, thus a copy of $\mathbb{R}$ as a sublattice. In the sequel, our basic structure on $C(X)$ will be that of "real lattice". Without loosing generality we may assume X is completely regular, Hausdorff and even realcompact (since $C(X)$ is lattice-ordered algebra unit preserving isomorphic to $C(\mu X)$).

At the crux of most attempts, if L^* denotes the bounded elements of a given real lattice L, then the following conditions somehow are needed: (a) L embeds into some $C(X)$, and (b) L^* is isomorphic to $C^*(X)$.

Under this general context, the unique contribution appearing in the literature for the more general case is that of Jensen [9] as a refinement that of Anderson [2], but by assuming richer compatible algebraic structures, namely for Φ-algebras. Our effort to weaken algebraic assumptions lead to characterizations of $C(X)$ as Riesz spaces [10, 8], real ℓ-groups [3] or semi-affine lattices [4]

However, the nicest and purely lattice-theoretic real lattices characterization of $C(X)$ for X a compact Hausdorff space is that of Anderson-Blair [1]. From this seminal result, and by adding convenient conditions we may characterize $C(X)$ as a real lattice [5] by constituting a generalization of the famous problem 81 of Birkhoff and Kaplansky (1948). In [6] we present a unified approach valid for any "convenient" subcategory of the real lattices.

By identifying the Samuel's compactification of a uniform space by means of certain partitions of a real lattice, and by setting equivalent conditions to equi-uniform continuity, a characterization of the real lattice $U(X)$ of real-valued uniformly continuous functions on a uniform space X is derived from [7].

References

[1] Anderson, F.W., Blair, R.L.: Representation of distributive lattices of continuous functions, Math. Ann. 143, 187–211 (1961).

[2] Anderson, F.W.: Approximation in systems of real-valued continuous functions. Trans. Amer. Math. Soc. 103, 249–271 (1962).

[3] Hušek, M., Pulgarín, A.: $C(X)$ as a real ℓ-group. Topology Appl. 157 (8), 1454–1459 (2010).

[4] Hušek, M., Pulgarín, : $C(X)$-objects in the category of semi-affine lattices, Appl. Categ. Structures, 19, 439–454 (2011).

[5] Hušek, M., Pulgarín, A.: $C(X)$ as a lattice: A generalized problem of Birkhoff and Kaplansky, Topolology Appl. 158, 904–912 (2011).

[6] Hušek, M., Pulgarín, A.: General approach to characterizations of $C(X)$. Topology Appl. 159, 1603–1612 (2012).

[7] Hušek, M., Pulgarín, A.: Lattices of uniformly continuous functions. To appear in Quaestiones Math.

[8] Hušek, M., Pulgarín, A.: On characterizing Riesz spaces $C(X)$ without Yosida representation. Positivity, DOI: 10.1007/s11117-012-0185-5.

[9] Jensen, G.A.: A note on complete separation in the Stone topology. Proc. Amer. Math. Soc. 21, 113–116 (1969).

[10] Montalvo, F., Pulgarín, A., Requejo, B.: Riesz spaces of real continuous functions. Positivity 14 (3), 473–480 (2010).