THE ZARISKI AND MARKOV TOPOLOGIES ON A GROUP

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Given a group G, Markov [4] defined an *elementary algebraic subset* X of G to be the solution-set of a one-variable equation over G, i.e.

$$X = \{x \in G \mid g_1 x^{\varepsilon_1} g_2 x^{\varepsilon_2} \cdots g_n x^{\varepsilon_n} = e_G\}$$

for $n \in \mathbb{N}$, $\varepsilon_1, \ldots, \varepsilon_n \in \mathbb{Z}$, and $g_1, \ldots, g_n \in G$. He also defined a subset X to be *unconditionally closed in* G if it were closed in every Hausdorff group topology on G.

These definitions implicitly introduced two T_1 topologies on G, now called the *Zariski topology* $\mathfrak{Z}_G$ of G, generated by the elementary algebraic subsets as closed sets, and the *Markov topology* $\mathfrak{M}_G$ of G, generated by the unconditionally closed subsets as closed sets (this topology is the intersection of every Hausdorff group topology on G, and it need not to be neither a group topology, nor even Hausdorff).

One can easily see that $\mathfrak{Z}_G \subseteq \mathfrak{M}_G$, and Markov himself proved that they coincide if the group G is countable. He asked whether these two topologies always coincide, and Hesse [5] built an example showing that this is not true in general. The authors of [2] and [6] indipendently proved that the equality $\mathfrak{Z}_G = \mathfrak{M}_G$ holds for groups $G = A \times \left(\bigoplus_{i \in I} G_i\right)$, where A is an abelian group and each group G_i is countable.

The more topologies the group G carries, the coarser $\mathfrak{M}_G$ is; while $\mathfrak{M}_G$ is itself a Hausdorff group topology if and only if it is contained in every Hausdorff group topology on G. In this case, $\mathfrak{M}_G$ is the minimum in the lattice of all Hausdorff group topologies on G. If this happens, the group G is called *algebraically minimal*. For example, it is a classic result [3] that, for the group $S(X)$ of the permutations of an infinite set X, the point-wise convergence topology is contained in every Hausdorff group topology on $S(X)$, so that $S(X)$ is algebraically minimal. Let $S_\omega(X)$ be the subgroup of $S(X)$ consisting of the finite-support permutations. It has been very recently proved in [1] that, for every subgroup G of $S(X)$, containing $S_\omega(X)$, $\mathfrak{Z}_G = \mathfrak{M}_G$ is the point-wise convergence topology on G. This extends Gaughan's result, and in particular all such subgroups are algebraically minimal.

References

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